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Statistical Description of Chaotic Attractors: The Dimension Function

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A method for the investigation of fractal attractors is developed, based on statistical properties of the distribution $P(\delta, n)$ of nearest-neighbor distances δ between points on the attractor. A continuous infinity of dimensions, called dimension function, is defined through the moments of $P(\delta, n)$. In particular, for the case of self-similar sets, we prove that the dimension function DF yields, in suitable points, capacity, information dimension, and all other Renyi dimensions. An algorithm to compute DF is derived and applied to several attractors. As a consequence, an estimate of nonuniformity in dynamical systems can be performed, either by direct calculation of the uniformity factor, or by comparison among various dimensions. Finally, an analytical study of the distribution $P(\delta, n)$ is carried out in some simple, meaningful examples.

KEY WORDS: Fractals; dynamical systems; nearest neighbors; Hausdorff dimension; uniformity of strange attractors.

1. INTRODUCTION

During the last few years, much of the attention in the study of dynamical systems has been paid to the onset of deterministic chaos.[(1)] Three general routes have been discovered and carefully described[(2)]: intermittency, period doubling, and quasiperiodicity transition. Only very recently have efforts also been addressed toward the understanding of chaotic attractors.

The relevant quantities characterizing such objects are fractal dimension[(3)] and metric entropy.[(4)] The former refers to static properties (invariant measure) and roughly estimates the number of independent variables involved in the process. The latter is a dynamic quantity which

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measures the loss of information on the initial conditions per unit time. As a matter of fact, a third set of relevant variables is represented by the Lyapunov exponents, which can, however, be easily computed only in numerically integrable cases, by using the method described in Ref. 5. Indeed, in any experiment, only a single variable time series is generally available. Thus, the attractor can be reconstructed by invoking the embedding theorem,[6] but a rigorous application of the technique in Ref. 5 still remains a difficult task and only some preliminary results have been obtained.[7]

Furthermore, it is worthwhile recalling the relations occurring between the three sets of quantities so far introduced. The metric entropy is, for instance, related to the sum of the positive Lyapunov exponents when invariant and Lebesgue measures are equivalent.[8] On the other hand, a relationship between dimension and Lyapunov exponents has been evidenced by Kaplan and Yorke.[9]

Metric entropy and fractal dimension can be used to advantage to discriminate between stochastic and purely chaotic systems. The fractal dimension, for instance, in the former case, is equal to the phase-space dimension d, while, in the latter, it is a smaller and, in general, noninteger number. Contrary to this, the commonly used power spectrum does not discriminate between the two cases which both yield a broad band structure.

In this paper, we focus on the problem of giving an accurate description of the scaling properties of a fractal set, when the observational resolution is increased. Namely, we introduce the probability distribution $P(\delta, n)$ of nearest-neighbor (nn) distances δ among n randomly chosen points on the attractor, and study the behavior of its moments $\langle \delta^\gamma \rangle$ in the large-n limit.[10] This allows definition of a "dimension function" $D(\gamma)$ which extends the semi-infinite Renyi hierarchy[11] to the whole real axis. In particular, when self-similarity is asymptotically established, we prove that the dimension function (DF) yields, for specific γ values, the capacity, information dimension, and all other Renyi dimensions. The main difference with respect to the Renyi approach, however, is that nonuniform covering is considered here.

We also show how knowledge of the whole DF is necessary to give a complete characterization of the fractal. As a consequence, the question of deciding which dimension is the most relevant turns out to be meaningless, while, instead, it is more relevant to focus our attention onto the spread among the different dimensions. For this purpose, we study the slope of the DF at the "fixed" point $\gamma = D(\gamma)$ as a first-order estimate of the nonuniformity of the set. This quantity has been previously named "uniformity factor,"[10] and its appellative is here better elucidated showing that it represents the growth rate of the relative variance of the distribution $P(\delta, n)$.

Finally, a direct study of $P(\delta, n)$ is performed both analytically, in some examples, and numerically.

The paper is organized as follows: In Section 2 we give a brief survey of the commonly used definitions of fractal dimension. In Section 3, the dimension function is introduced, and its relationship with capacity and Hausdorff dimension discussed. The uniformity factor is also introduced in terms of a suitable entropy. In Section 4, the exact relation linking the DF to the Renyi hierarchy is derived. In Section 5, the expression of the probability distribution $P(\delta, n)$ is explicitly computed for uniform Cantor sets[3] on a line, and numerically evaluated for two iterative maps. In Section 6, the numerical method for determination of the DF is explained, and results are given both for maps and flows. In Appendices A and B, two special cases are analyzed, while an integral expression for $P(\delta, n)$ in a binary Cantor set is obtained in Appendix C.

2. DEFINITIONS AND PREVIOUS METHODS

Since the beginning of the century, many mathematicians have been engaged in the task of a proper geometrical characterization of a fractal set. As a consequence, the attempt to satisfy the most general validity requirements led to the introduction of many definitions of dimension which, frequently, differ only in very subtle aspects. They can be roughly classified into two groups, the first one deriving from purely geometrical requests, the second being related to information theory.

The definitions of the first group usually seem to give identical results for physical systems. Therefore, we refer to Mandelbrot's[(3)] book for a fairly complete review of the subject, and here we only recall the most common of them, briefly discussing the possibility of numerical applications.

We start from the capacity, whose definition naturally derives from the concept of uniform covering of a given set. Namely, after having covered the set with $N(\varepsilon)$ balls of radius ε, we introduce the "γ volume"

$$\bar{L}_\gamma(\varepsilon) = \varepsilon^\gamma \inf N(\varepsilon) \tag{2.1}$$

where the infimum is taken over all possible coverings. The capacity D_0 is

[3] Throughout the paper, a Cantor set is called "uniform" whenever it is generated by keeping equal intervals with equal weights. This is only a particular case of a "uniform attractor" as defined in the text.

then defined either as the infimum over all γ's such that $\bar{L}_\gamma(\varepsilon)$ shrinks to 0 for the vanishingly small ε, or, equivalently, as

$$D_0 = -\lim_{\varepsilon \to 0} \frac{\ln \inf N(\varepsilon)}{\ln \varepsilon} \tag{2.2}$$

Some variants of this definition exist[3] which are almost equivalent. The box-counting algorithm, for instance, is based on the introduction of a uniform partition of size ε in the phase space, and then on the computation of the number $N'(\varepsilon)$ of nonempty cells. Owing to the nonoverlapping property of these boxes, the quantity $N'(\varepsilon)$ is not generally significantly different from $\inf N(\varepsilon)$, and the two definitions can be considered equivalent. The limitations of a numerical application of this method have already been investigated[12,13] and are mainly due to a very slow statistical convergence. In fact, the estimation of $N'(\varepsilon)$ requires generation of an increasing number of points on the attractor verifying whether they fall insides boxes already visited. A reliable value of N', however, can only be achieved after an enormously large number of points, thus rendering the method unfeasible for a phase-space dimension larger than 2.

A refinement of the concept of covering has been proposed by Hausdorff,[14] who considered balls of variable size ε_i, with the constraint $\varepsilon_i \leqslant \varepsilon$. As a consequence, the γ volume turns out to be straightforwardly generalized to

$$L_\gamma(\varepsilon) = \inf \sum_i \varepsilon_i^\gamma \tag{2.3}$$

where the infimum is taken over all coverings satisfying $\varepsilon_i \leqslant \varepsilon$. In analogy with the definition of capacity, the Hausdorff dimension D_H is, hence, the infimum over all γ's such that $L_\gamma(\varepsilon)$ shrinks to zero. Even when it is possible to build several examples showing that $D_H \neq D_0$,[4] it is still not clear whether there is any relevant difference in physical systems.[6] Finally, let us recall that, as for the capacity, some variants to D_H have been introduced (see, for instance, Ref. 16).

A physical approach to the study of chaotic attractors is, instead, naturally related to the scaling properties of the probability density (invariant measure), when the observational resolution is increased. Farmer, Ott, and Yorke,[17] and Grassberger and Procaccia,[18] following this

[4] The Hausdorff dimension of any countable set of points can be proved to be 0.[15] Hence, for example, the set made of all rational numbers within the interval [0, 1] has $D_H = 0$, while the capacity is easily shown to be 1.

approach, consider the mass $\mu(\mathbf{x}, \varepsilon)$ contained in a ball of radius ε, centered around $\mathbf{x}$, as the relevant variable to be studied.

Under the assumption that $\mu(\mathbf{x}, \varepsilon)$ scales as $\varepsilon^{D_p(\mathbf{x})}$, the pointwise dimension[17] is defined as

$$D_p(\mathbf{x}) = \lim_{\varepsilon \to 0} \frac{\ln \mu(\mathbf{x}, \varepsilon)}{\ln \varepsilon} \tag{2.4}$$

An average over all points is usually performed to get a more reliable estimate. If, instead, the quantity to be averaged is $\mu^2(\mathbf{x}, \varepsilon)$ the correlation-integral exponent ν[18] will be found,

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \Theta(\varepsilon - |\mathbf{x}_i - \mathbf{x}_j|) \xrightarrow[\varepsilon \to 0]{} \varepsilon^\nu \tag{2.5}$$

where Θ is the Heavyside step function and $\mathbf{x}_i$, $\mathbf{x}_j$ are randomly chosen. These two exponents belong to a class of generalized dimensions introduced by Renyi[11] in terms of suitable entropies $K_q(\varepsilon)$, $q = 0, 1, 2,..., n$,

$$\begin{aligned} K_q(\varepsilon) &= \frac{1}{1-q} \ln \sum_{i=1}^{N(\varepsilon)} P_i^q(\varepsilon), \qquad && q \neq 1 \\ K_1(\varepsilon) &= -\sum_{i=1}^{N(\varepsilon)} P_i(\varepsilon) \ln P_i(\varepsilon), \qquad && q = 1 \end{aligned} \tag{2.6}$$

assuming a uniform partition of the phase space with boxes of size ε, and defining $P_i(\varepsilon)$ as the probability of the ith box. Accordingly, the Renyi dimensions are given by

$$D_q = -\lim_{\varepsilon \to 0} \frac{K_q(\varepsilon)}{\ln \varepsilon} \tag{2.7}$$

In particular, D_0 is the capacity, while D_1 is called information dimension (see also Ref. 19), and usually coincides with the pointwise dimension. Moreover the dimension D_2 is nothing but the exponent ν, and the general inequality $D_q \geq D_p$ (when $p > q$) holds.[20]

A sort of dual approach has been adopted in Ref. 21, but that fractal dimension is not univocally defined.[22] For other exponents and computational methods, see Refs. 23, 17, and references therein.

3. THE DIMENSION FUNCTION

As we have seen in the previous section, many different definitions of dimension are available for a characterization of a strange attractor. Since it appears disputable to establish which of them is physically most relevant, we have preferred to follow a different strategy. Instead of studying a single quantity which gives only average information on the attractor, we consider an infinite class of dimensions (all easily computable), exactly in the same way as different moments of a probability distribution are used to characterize a statistical ensemble.

Let S be a bounded set in a d-dimensional Euclidean space E. The entire information on the fractal features of S is carried by the scaling properties of the density probability, as the observational resolution is increased. Here, instead of studying the properties of the invariant measure, we find it fruitful to follow a different approach. Namely, we start considering a reference point $\mathbf{x}$ plus $(n-1)$ others, all of them chosen at random with respect to the natural measure, on the attractor. We now define $\delta(n)$ as the distance between $\mathbf{x}$ and its nearest-neighbor $\mathbf{y}$ among the $(n-1)$ other points, and introduce the probability distribution $P(\delta, n)$ of *nn* distances among n points. A first rough connection of $P(\delta, n)$ with the fractal dimension is evidenced by looking at its first moment $\langle \delta \rangle \equiv M_1(n) \equiv \int \delta P(\delta, n)\, d\delta$, which asymptotically can be argued to depend on n as

$$\langle \delta \rangle \equiv M_1(n) \sim n^{-1/D} \tag{3.1}$$

where D is a suitable dimension whose relation with the other definitions given in the previous section will be clarified later.

Now, following Ref. 10, we naturally extend (3.1) to the generic moment of order γ,

$$\langle \delta^\gamma \rangle \equiv M_\gamma(n) \equiv \int_0^\infty \delta^\gamma P(\delta, n)\, d\delta = K n^{-\gamma/D(\gamma)} \tag{3.2}$$

where $D(\gamma)$ is a γ-dependent definition of dimension hereafter called dimension function (DF),

$$D(\gamma) = -\lim_{n \to \infty} \frac{\gamma \ln n}{\ln M_\gamma(n)} \tag{3.3}$$

The prefactor K, on the other hand, depends on both γ and n. However, its dependence on γ is, by definition, irrelevant, while the dependence on n reduces, in a large class of systems, to an unessential periodicity in $\ln n$.[24]

The role of γ is to enhance or depress different δ-length scales. More precisely, for increasing (decreasing) γ, the larger (smaller) distances are more weighted. As a consequence, from the definition (3.3), it is possible to argue that $D(\gamma)$ is a monotonic nondecreasing function of γ, since the larger distances must decrease more slowly. A rigorous proof will be shown later.

To investigate the meaning of the DF, we start with an analytical study of a simple but meaningful example: the nonuniform Cantor set,[5] which is defined as follows. Starting from the initial segment (0, 1), an internal open segment (central and equal to 1/3 for the standard Cantor set) is deleted, and two closed segments ($\langle 1 \rangle$ and $\langle 2 \rangle$) of lengths α_1^{-1} and α_2^{-1}, respectively, are left. By infinitely iterating this procedure, a fractal set is finally generated. Its capacity, by simple scaling arguments, can be proved to satisfy the following implicit relation[17]:

$$1 = \alpha_1^{-D_0} + \alpha_2^{-D_0} \tag{3.4}$$

As far as one is interested only in purely geometrical aspects, (3.4), specifying the dimension of the support is sufficient to describe the set. However, a more general model of attractor requires the attribution of probability weights p_1 and p_2 ($p_1 + p_2 = 1$) to the contraction rates α_1 and α_2, respectively. Indeed, a complete identification of any point can be achieved by means of an infinite series of "bits"specifying whether the point belongs to the right or left part, at each resolution level. Evidently, these coefficients do not alter the geometrical structures of the support and, hence, the capacity is not affected either. All the other Renyi dimensions $D_q (q > 1)$ containing statistical information on the attractor, depend instead on weights p_1, p_2.

The self-similarity properties of this nonuniform Cantor set can be exploited to obtain a simple relation for the probability $P(\delta, n)\, d\delta$. Indeed, by noting that the two subsets $\langle 1 \rangle$, $\langle 2 \rangle$ are rescaled versions of the whole set, with space and population scale factors α_1, α_2 and p_1, p_2, respectively, the following relation holds:

$$P(\delta, n)\, d\delta = p_1 P(\alpha_1 \delta, p_1 n)\, \alpha_1 d\delta + p_2 P(\alpha_2 \delta, p_2 n)\, \alpha_2 d\delta \tag{3.5}$$

in the limit of large n's, when the probability that the nn of a point x does not belong to the same subset, is negligible. Now, from (3.5), a relation for the moments is easily derived,

$$M_\gamma(n) = p_1 \alpha_1^{-\gamma} \int_{\langle 1 \rangle} y^\gamma P(y, p_1 n)\, dy + p_2 \alpha_2^{-\gamma} \int_{\langle 2 \rangle} y^\gamma P(y, p_2 n)\, dy \tag{3.6}$$

and, recalling the definition (3.2), without the unessential factor K, an implicit exact relation for the DF is found,

$$1 = \alpha_1^{-\gamma} p_1^{1-\gamma/D(\gamma)} + \alpha_2^{-\gamma} p_2^{1-\gamma/D(\gamma)} \tag{3.7}$$

[5] It corresponds to a horizontal section of the generalized Baker transformation.[17]

We first discuss such a result in the simplified case $p_1 = p_2 = 1/2$, when (3.7) can be simply solved for $D(\gamma)$[10]:

$$D(\gamma) = \gamma \frac{\ln 2}{\ln 2 - \ln(\alpha_1^{-\gamma} + \alpha_2^{-\gamma})} \tag{3.8}$$

By comparing (3.4) and (3.8), it is readily seen that $\gamma = D_0$ yields the fixed-point equation $D(D_0) = D_0$, which is an implicit relation for the capacity. Such a preliminary result is reinforced by looking at the more general equation (3.7), from which it turns out that, whenever $\gamma = D(\gamma)$, (3.4) is recovered, independently of the weights p_1, p_2. Hence, we can conclude that the request to satisfy the fixed-point relation $D(\gamma) = \gamma$ is sufficient to determine the capacity. Moreover, this example is easily extended to any attractor composed of m subsets, embedded in a d-dimensional Euclidean space, which is representative of a large class of physical systems.[25]

We can now formally interpret the general equation (3.3) as a recursive relation

$$\gamma_{k+1} \equiv D(\gamma_k) = -\lim_{n \to \infty} \gamma_k \frac{\ln n}{\ln M_{\gamma_k}(n)} \tag{3.9}$$

Starting with a trial initial value γ_0, a first approximation γ_1 of the capacity is computed and then used as a new γ coefficient in (3.9) to obtain a second-order approximation. The existence of at least a fixed point of Eq. (3.9) is quite plausible, and we simply assume it without further comment.

It then becomes important to perform a linear stability analysis of the fixed point to prove the convergence of the procedure described above. Still referring to the nonuniform Cantor set, we must simply take the derivative of (3.7) with respect to γ and then specialize the solution for $\gamma = D_0$,

$$D'(D_0) = 1 + D_0 \frac{\alpha_1^{-D_0} \ln \alpha_1 + \alpha_2^{-D_0} \ln \alpha_2}{\alpha_1^{-D_0} \ln p_1 + \alpha_2^{-D_0} \ln p_2} \tag{3.10}$$

It is easy to verify that $0 \leqslant D' < 1$ always except for the unphysical cases p_1 or $p_2 = 0$, when $D' = 1$. Hence, the fixed point is stable and the recurrence (3.9) converges. A more general study of (3.7) shows that the DF is a monotonic nondecreasing function for any choice of the parameters and is bounded between two horizontal asymptotes. The upper one, $D(\infty) = \max(-\ln p_1/\ln \alpha_1, -\ln p_2/\ln \alpha_2)$, is the dimension of a uniform Cantor set generated by keeping $1/p_1(1/p_2)$ segments of length $1/\alpha_1(1/\alpha_2)$. Analogously, $D(-\infty) = \min(-\ln p_1/\ln \alpha_1, -\ln p_2/\ln \alpha_2)$ can be interpreted in the same way. Incidentally, the two limits correspond to the slowest and fastest contraction rates towards zero of the *nn* distances.

When $D(-\infty)$ coincides with $D(\infty)$, i.e., $p_i = \alpha_i^{-D_0}$ $(i = 1, 2)$, the DF is everywhere constant and we speak of uniform attractor (see Ref. 25).

Reverting to a general dynamical system, we now investigate the relationship of definition (3.3) with Hausdorff dimension and capacity. It is useful to rewrite (3.2) in a less rigorous, but more transparent, way. In fact, we substitute the integral in (3.2) by a sum over the n points, all of them chosen as reference points, evaluating the *nn* distances $\delta_i(n)$ $(i = 1,..., n)$,

$$M_\gamma(n) = \frac{1}{n} \sum_{i=1}^{n} \delta_i^\gamma(n) \tag{3.11}$$

Since the definition of $D(\gamma)$ holds for large n, the inaccuracy of (3.11) can be arbitrarily reduced, by choosing n sufficiently large. After multiplying (3.2) by n and using definition (3.11), a quantity $\tilde{L}_\gamma(n)$ is introduced, analogously to the $L_\gamma(\varepsilon)$ of (2.3),

$$\tilde{L}_\gamma(n) \equiv \sum_{i=1}^{n} \delta_i^\gamma(n) \sim n^{1-\gamma/D(\gamma)} \tag{3.12}$$

In the limit $n \to \infty$, an immediate parallel emerges with the definitions of D_0 and D_H given in Section 2. In fact, $\tilde{L}_\gamma(n)$ diverges whenever $\gamma < D(\gamma)$, and vanishes for $\gamma > D(\gamma)$. The fixed point $\gamma = D(\gamma)$ finally coincides with the capacity, as discussed further in the next section. The γ-volume $\tilde{L}_\gamma(n)$ is not necessarily constant at the fixed point, where logarithmic corrections might show up (see Appendix A for an example where they are present). An important point to discuss next is the uniqueness of the solution $D(\gamma) = \gamma$. In fact, in some very special cases, even an infinite number of solutions can be found (see Appendix A). The consequent ambiguity can be avoided anyhow by taking the infimum over all γ's satisfying $D(\gamma) = \gamma$,

The parallel with D_H can be pushed even further if we interpret the δ_i's as the diameters of "δ neighborhoods" around the n points, and prove that such balls do indeed constitute a covering of the attractor. First, notice that there is no overlapping of balls. Next, we require that, once given n balls, all the points of the attractor fall within them. If the set is uniform, i.e., the mass in the ith ball scales as $P_i \propto \delta_i^{D_0}$, the uncovered fraction will vanish in the limit $n \to \infty$: In fact, the left-hand side of (3.12) becomes a sum of probabilities P_i. In the general nonuniform case, provided that self-similarity asymptotically holds, we reasonably expect to be able to cover a constant fraction f of the D_0-dimensional volume. To test this conjecture, we computed f, as a function of n, in two cases: The generalized Baker transformation and the Sinai map.[26] The first map is defined by

$$\begin{aligned}
x_{n+1} &= x_n/3, & 0 < y_n < a \\
y_{n+1} &= y_n/a, & \\
x_{n+1} &= (1+x_n)/3, & a < y_n < 1-a \\
y_{n+1} &= (y_n - a)/(1-2a), & \\
x_{n+1} &= (2+x_n)/3, & 1-a < y_n < 1 \\
y_{n+1} &= (y_n - 1 + a)/a &
\end{aligned} \tag{3.13}$$

A horizontal section of its asymptotic solution is a Cantor set with $\alpha = 3$, a nonzero probability p_2 in the middle part $(1/3 < x < 2/3)$, and equal weights $p_1 = p_3$ in the side stripes. Hence, its capacity is $D_0 = 2$ and the degree of nonuniformity is controlled through a [$a = 1/3$ corresponding to a flat probability distribution $Q(x, y)$]. In Fig. 1, we report $f(n)$ as a function of n for $a = 0.1, 0.2$. Notice that, after some initial oscillations, the fraction f goes to a constant value, which gets closer to 1 for a tending to 1/3. The same behavior has been observed in the case of the Sinai map [see later, Section 6, (6.1)], with $g = 0.3$, a parameter value which corresponds to a highly nonuniform attractor (with $D_0 = 2$).

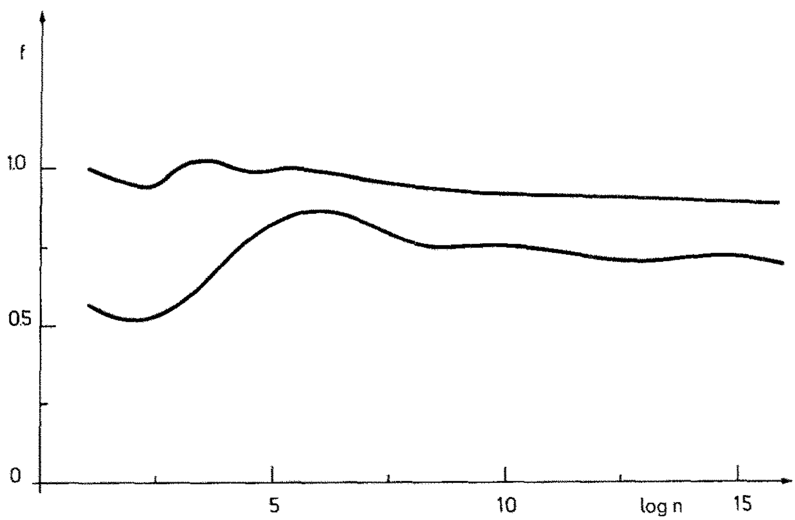

Fig. 1. Covered fraction f of the support of the attractor, for the generalized Baker transformation (3.13), as a function of $\log(n)$. The upper curve refers to $a = 0.2$, and the lower one to $a = 0.1$.

Now, since the fixed point has been clarified to be the power of γ which allows $\tilde{L}_\gamma(n)$ to converge to a finite value for large n, $\tilde{L}_\gamma(\infty)$ itself can be interpreted as a generalization of the volume. The main difference with respect to D_H, comes instead from the kind of covering adopted. While the Hausdorff dimension requires the infimum over all possible coverings, here we have chosen a particular one: Indeed, in Appendix A, we show, in a special case, that we measure D_0 and not D_H.

Let us finally discuss the general shape of the DF, and the stability of the fixed point. Taking the derivative of both sides of (3.3) with respect to γ, we obtain, in the large-n limit, the following expression:

$$D'(\gamma) = \frac{D^2(\gamma)}{\gamma^2}\left[1 - \frac{H_n(\gamma)}{\ln n}\right] \tag{3.14}$$

The function $H_n(\gamma)$ is an entropy defined by

$$H_n(\gamma) = -\sum_i p_i \ln p_i \tag{3.15}$$

with $p_i = \delta_i^\gamma / \sum_i \delta_i^\gamma$. This "$\delta$ entropy" is a positive-definite quantity which can always be written as

$$H_n(\gamma) = \sigma(\gamma) \ln n \tag{3.16}$$

where $0 \leqslant \sigma(\gamma) \leqslant 1$. The value $\sigma(\gamma) = 1$ corresponds to a uniform attractor. The extreme situation $\sigma(\gamma) = 0$ corresponds to unphysical cases, where at least one $\delta_i(n)$ does not vanish (nonrecurrent point). By substituting (3.16) in (3.14), we obtain an asymptotic expression, independent of n,

$$D'(\gamma) = \frac{D^2(\gamma)}{\gamma^2}\,[1 - \sigma(\gamma)] \tag{3.17}$$

Therefore, being $D'(\gamma) \geqslant 0$, the DF is monotonic nondecreasing. Moreover, for $\gamma = D_0$, we obtain the uniformity factor

$$\lambda \equiv D'(D_0) = 1 - \sigma(D_0) = 1 + \frac{\sum_i \delta_i^{D_0} \ln \delta_i^{D_0}}{\sum_i \delta_i^{D_0} \ln n} \tag{3.18}$$

which, of course, lies within 0 and 1, thus generally proving the stability of the fixed point. In particular, for a uniform attractor, $\lambda = 0$ (superstability). An exception, where $\lambda = 0$, but the set is nonuniform, is treated in Appendix B. Finally, note that λ is directly computable by its definition (3.18).

4. RELATION WITH THE RENYI DIMENSIONS

In the previous section, we have seen that the fixed point of the DF yields the capacity D_0. Now, we study the connection between the DF and all the other Renyi dimensions as defined in Section 2.

We again start, as in Section 3, with the nonuniform Cantor set, whose subsets, $\langle 1 \rangle$ and $\langle 2 \rangle$, have probability weights p_1 and p_2, respectively. By considering the attractor as being composed of two parts, and from definition (2.6) of Renyi entropies, we have

$$e^{(1-q)K_q(\varepsilon)} = \sum_{i=1}^{N(\varepsilon)} P_i^q(\varepsilon) = \sum_{\langle 1 \rangle} P_i^q(\varepsilon) + \sum_{\langle 2 \rangle} P_i^q(\varepsilon) \tag{4.1}$$

By normalizing the probabilities in each of the two subsets and noting that

$$\sum_{\langle 1 \rangle} P_i = p_1, \qquad \sum_{\langle 2 \rangle} P_i = p_2 \tag{4.2}$$

we obtain

$$e^{(1-q)K_q(\varepsilon)} = p_1^q \sum_{\langle 1 \rangle} e^{(1-q)K_q(\alpha_1 \varepsilon)} + p_2^q \sum_{\langle 2 \rangle} e^{(1-q)K_q(\alpha_2 \varepsilon)} \tag{4.3}$$

each of the two subsets being a rescaled version of the whole attractor. Finally, by recalling the definition of D_q, we get the following expression (see also Refs. 20 and 25):

$$1 = p_1^q \alpha_1^{-(1-q)D_q} + p_2^q \alpha_2^{-(1-q)D_q} \tag{4.4}$$

By comparing (4.4) with (3.6) we obtain the relation

$$D[\gamma = (1-q)\, D_q] = D_q \tag{4.5}$$

Again (4.5) has a general validity under the assumption that self-similarity is established at least for large n. This can be easily proven by considering an initial volume and keeping m, instead of two, subsets and repeating the procedure discussed above.

In particular, when $q = 0$, it is readily seen that we recover the fixed-point relation (3.9). For $q \to 1$, the following important relation is obtained:

$$D_1 = \lim_{\gamma \to 0} D(\gamma) = -\frac{n \ln n}{\sum_i \ln \delta_i(n)} \tag{4.6}$$

yielding the information dimension D_1 in explicit form which, for the Cantor set, turns out to be

$$D(0) = -\frac{p_1 \ln p_1 + p_2 \ln p_2}{p_1 \ln \alpha_1 + p_2 \ln \alpha_2} \tag{4.7}$$

Moreover, by a recursive procedure, analogous to that described for the capacity, it is possible to obtain all the other Renyi dimensions. The relation (4.5) is illustrated in Fig. 2 where we can see how the D_q's correspond to the crossing points of the DF with a series of straight lines with slopes $1/(1-q)$. The information dimension D_1, being its abscissa ($\gamma = 0$) known *a priori*, is the simplest one to be measured without iterative methods. We can also see how $D(0)$ is close to the pointwise dimension.(17) Indeed, starting from the idea that the nn distances around a given point $\mathbf{x}$ on the attractor scale as $n^{-1/D(\mathbf{x})}$, we simply use the average in (4.6) to obtain a reliable estimate of the local dimension. The choice of δ as the relevant scaling quantity, instead of the mass as in Ref. 16, should be essentially irrelevant.

While the Renyi dimensions extend to the left of D_0 (region L), here the right part (R) is naturally defined, too. Region R in general cannot be

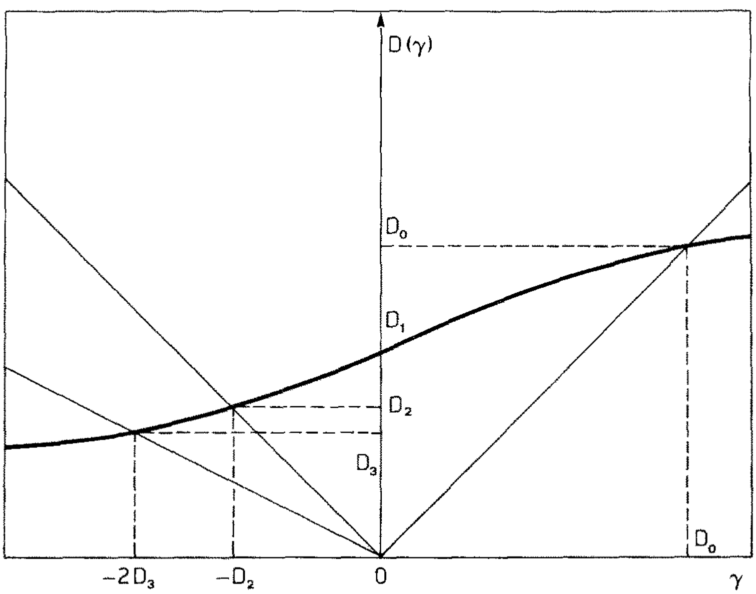

Fig. 2. Sketch of the DF $D(\gamma)$ versus γ displaying a geometrical picture of relation (4.5). The various intersections define the Renyi dimensions D_0, D_1, D_2, etc.

thought to represent other meaningful dimensions, as its values can increase even above the phase-space dimension d, as shown in Appendix A. Of course, this kind of example is never encountered in physical attractors, which do not contain isolated points. However, even disregarding pathological cases, it is always possible to imagine situations where the distances δ shrink to zero slower than $n^{-1/d}$ in some particular subregions. The nonuniform Cantor set, for instance, displays such behavior when p_1 is chosen so small that $-\ln p_1/\ln \alpha_1 > 1$.

Anyhow, R contains further information on the attractor, and this is particularly evidenced in the example of Appendix A, where the right part exhibits a completely different behavior from that of the left part. More generally, choosing δ^{D_0} as the new independent variable, R is straightforwardly related to the moments of the probability distribution $W(\delta^{D_0}, n)$,[6] In particular, the first moment goes as

$$M_{D_0} \sim 1/n \tag{4.8}$$

while the second one is simply M_{2D_0}. Hence, the relative r.m.s.

$$\Delta(n) = \frac{(M_{2D_0} - M_{D_0}^2)^{1/2}}{M_{D_0}} \tag{4.9}$$

behaves as

$$\Delta(n) \sim n^{1 - D_0/D(2D_0)} \tag{4.10}$$

since, for large n's, $M_{D_0}^2 \ll M_{2D_0}$ owing to $D(2D_0) > D_0$. In the linear approximation $D(2D_0) = D_0 + D'(D_0)\, D_0$, which is widely justified (see Section 6),

$$\Delta(n) \sim n^{\lambda} \tag{4.11}$$

where we recall that $\lambda = D'(D_0)$ is the "uniformity factor" introduced in the previous section. Equation (4.11) now motivates the appellative given, as it measures the rate of broadening of the distribution $W(\delta^{D_0}, n)$ for increasing n.

5. PROBABILITY DISTRIBUTION OF *nn* DISTANCES

In this section, we investigate the global features of $P(\delta, n)$ for fixed n. We first start from the simplest example of a uniform Cantor set with

[6] The probability $W(\delta^{D_0}, n)$ is, essentially, the same as $P(\delta, n)$: The change of name is due to the change of the first argument.

$p_1 = p_2 = 1/2$ and $\alpha_1 = \alpha_2 = \alpha$. In such a case, the self-similarity relation (3.5) reads as

$$P(\delta, n) = \alpha P(\alpha\delta, n/2) \tag{5.1}$$

It is worthwhile introducing the probability $S(\delta, n)$ of a reference point x to have a nearest neighbor, chosen among $(n-1)$ points, within a distance δ,

$$S(\delta, n) = \int_0^\delta P(y, n)\, dy \tag{5.2}$$

which is both more manageable and satisfies a simpler relation than (5.1), namely,

$$S(\delta, n) = S(\alpha\delta, n/2) \tag{5.3}$$

This functional relation straightforwardly implies

$$S(\delta, n) = F(\delta^{\ln 2/\ln \alpha} n) = F(\delta^{D_0} n) \tag{5.4}$$

where F is a generic function of the argument $n\delta^{D_0}$ which, hence, contains the scaling relation between δ and n defining the dimension D_0. The complete determination of $S(\delta, n)$ now requires some boundary conditions (i.e., the dependence on δ for a fixed n or vice versa).

We prefer to follow a direct method which allows for the evaluation of $S(\delta, n)$ for any n and not only for large n, as requested for the validity of (5.4). In fact $S(\delta, n)$ is simply the probability that, once a reference point x has been chosen, at least one out of the other n points is closer than δ, i.e., $S(\delta, n)$ is complementary to the probability that no points fall within a distance δ from x. Hence, exploiting the uniformity of the probability density [i.e., the mass inside a segment length 2δ is $(2\delta)^{D_0}$], $S(\delta, n)$ turns out to be

$$S(\delta, n) = 1 - [1 - (2\delta)^{D_0}]^n \tag{5.5}$$

which holds for any value of n. To make a comparison with the scaling relation (5.4), we notice that, for large n's, the distances δ are small and (5.5) can be approximated by using the definition of the Neper's number e,

$$S(\delta, n) = 1 - \exp[-n(2\delta)^{D_0}] \tag{5.6}$$

This relation satisfies (5.4), as it should, being the last one defined for large n's. Now, taking the derivative of (5.6) with respect to δ, an explicit expression for $P(\delta, n)$ is obtained,

$$P(\delta, n) = 2D_0 n(2\delta)^{D_0 - 1} \exp[-n(2\delta)^{D_0}] \tag{5.7}$$

and we see that, for $\delta \to \infty$, $P(\delta, n)$ tends to 0 as $\exp[-n(2\delta)^{D_0}]$, while for $\delta \to 0$ it diverges owing to the inequality $D_0 < 1$. Expression (5.7) can be recognized as a Brody distribution[27]; in particular, in the nonfractal case $D_0 = 1$, an exponential is recovered.

Let us now perform the analysis of a simple nonuniform set, namely, the binary Cantor set defined by $\alpha = 2$ and $p_1 \neq p_2$. Once chosen at random an interval of width 2δ around the point x, the mass inside that segment is now dependent on x. Indeed, if we divide the interval $(0, 1)$ into 2^k equal segments (of width $2\delta = 2^{-k}$), any of them being fully identified by a sequence of k bits, their respective weight will be given by the probability of a suitable ratio of 0's and 1's. Hence, the evaluation of $S(\delta, n)$ requires an average of (5.5) over all sequences of bits to be performed, and this causes the uniformity factor to be different from 0. A detailed calculation is carried out in Appendix C.

To complete the discussion on the nn distribution, we have numerically evaluated $W(\delta^{D_0}, n)$ for the attractors of Sinai [see, later, equation (6.1)], Henon,[28] and Zaslavskij.[29] The choice of W, instead of P, is motivated by the fact that, in this way, we always obtain an exponential distribution for uniform attractors. In Fig. 3, we report $\ln W(\delta^{D_0}, n)$ versus δ^{D_0}, for $n = 2^{12}$, in the above-mentioned cases. Note that the deviations from a straight-line behavior, present in the Henon and Sinai attractors, indicate their nonuniformity.

From the above results, the advantage of transferring the definition of dimension from the phase-space probability density to the nn-distances probability $P(\delta, n)$ clearly emerges.[7] First, this allows us to work with scalars (δ) instead of vectors, thus simplifying the numerical computations. Second, the method guarantees a constant-fraction covering of the support, thus ensuring the reliability of our results.

6. NUMERICAL METHODS AND RESULTS

6.1. Numerical Methods

The most direct application of (3.2) is achieved by storing an array containing a large number of points N, chosen as an integer power of $2 (N = 2^k)$, in a sequential way with an appropriate delay. The mean value $\langle \delta(n) \rangle$, with $n = 2^j$, is computed by dividing the N data into $2^{(k-j)}$ blocks, each containing n points. The distances $\delta_i(n)$ are evaluated by comparing the point $\mathbf{x}_i$ with all the others contained in the same block. The $\delta_i(n)$ are then stored in an array, and the average is performed over all the N points:

[7] Methods of Refs. 18, 19, 21, and 23 offer similar advantages.

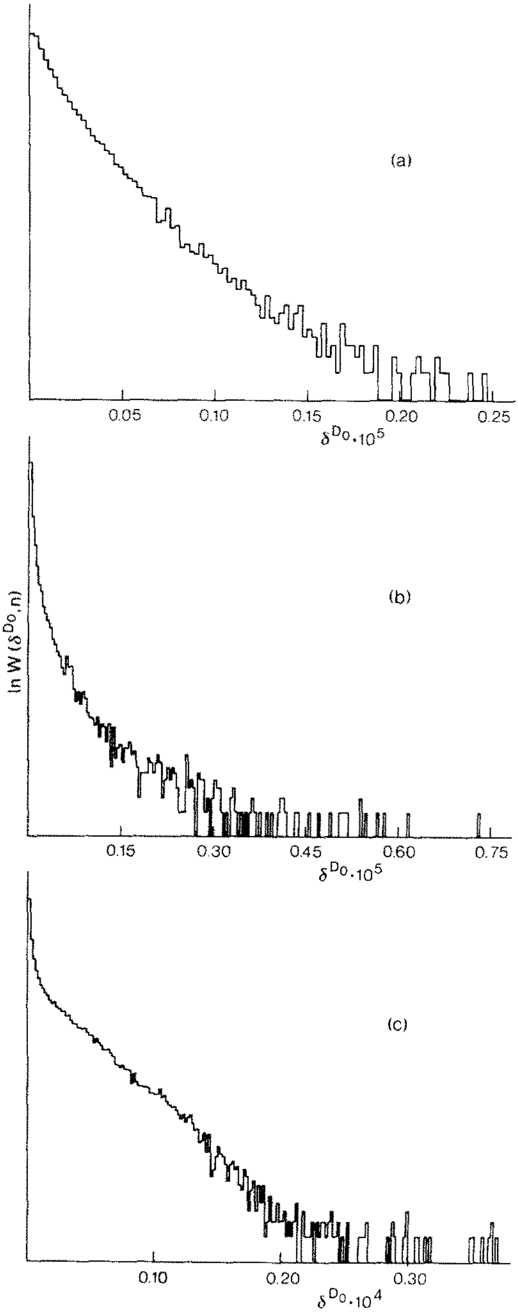

Fig. 3. Plot of ln $W(\delta^{D_0}, n)$ versus δ^{D_0}, with $n = 2^{12}$, for (a) Sinai map ($g = 0.3$), (b) Henon attractor, and (c) Zaslavskij attractor.

In this way, independently of n, the same statistical weight is attributed to the values $\langle \delta(n) \rangle$. Such a procedure, however, does not allow us to reach very large values of n. This limitation may prevent correct estimations, especially when the self-similarity shows up only for very small length scales (large n).[24] Therefore, we have sometimes preferred to employ a less rigorous, but equally reliable method: A fixed number m of points is chosen at random on the attractor and stored in the computer memory. They are then compared with every newly generated point to calculate the quantities $\delta_i(n)(i=1,..., m)$, which are continuously updated in an additional array. The average (3.2) is then performed at given n values (exponentially spaced), over the m points.

In the present paper, we have usually chosen $m=5000$, so that it has been possible to attain $n=2^{19}$ points in a reasonable amount of CPU time on an IBM 3083 computer. Apart from time limitations, another difficulty arises from the divergency of the relative r.m.s. $\Delta(n)=n^{\lambda}$ [Eqs. (4.9)–(4.11)]. In fact, the number m may result in being too small to guarantee a satisfactory statistical accuracy when increasing n.

For maps, the time needed to generate the points is negligible in comparison to that spent in calculating the nn distances, for five- to seven-

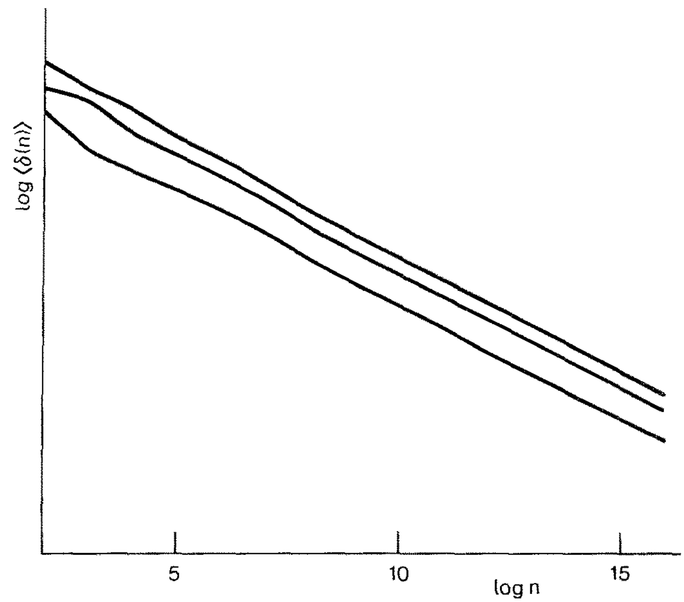

Fig. 4. Typical curves $\log\langle \delta(n) \rangle$ versus $\log n$ for first, second, and third nearest neighbors, respectively, from bottom to top.

dimensional flows, the two times are of the same order. Once the $\delta_i(n)$ are known, an estimate of $D(\gamma)$ is possible for many values of γ simultaneously, without much effort, thus getting a fairly good picture of the attractor.

Special care must be devoted to renormalize all variables between 0 and 1, in order not to underestimate the contribution of those having a limited range of variation. This is a numerical procedure which avoids the necessity of using a very large number of points (small distances).

Another important comment concerns the order of the neighbors. It is, indeed, easy to verify that the relations discussed so far in terms of the first nn also hold for higher-order nn, up to unessential multiplicative factors. However, the sensitiveness to statistical fluctuations is much weaker for second and third nn, rather than for first nn (see Fig. 4). Hence, this property allows more accurate estimations, at the expense of larger memory requirements only. We have always worked with third nearest neighbors.

Finally, a few comments on the integration algorithm adopted for the differential equations. We have used a modified fourth-order Runge–Kutta–Gills method with integration steps between 10^{-2} and 10^{-3}, depending on the system.

6.2. Results

In Table I, we report the dimension function $D(\gamma)$, computed at five different values of γ, for the Henon attractor,[28] the Zaslavskij attractor,[29] the Sinai map,[26] and two flows: The Roessler hyperchaos system[31] and a seven-mode truncation of the Navier–Stokes equations.[32] We have always averaged over 5000 points letting n go to 2^{19} for the maps, and 2^{16} for the flows. In the case of the Sinai map and the Roessler system, a measure was also performed with $40\,000 \times 2^{16}$ points and $40\,000 \times 2^{15}$ points, respectively. The DF is easily computed for the Henon map, as the uniformity factor is not very large, even though this attractor cannot be considered uniform. The Zaslavskij attractor, on the contrary is quite uniform, but care must be taken because the "self-similarity period"[24] is very long.

The Sinai map,

$$\begin{aligned} x_{n+1} &= x_n + y_n + g\cos(2\pi y_n) && \mod 1 \\ y_{n+1} &= x_n + 2y_n && \mod 1 \end{aligned} \tag{6.1}$$

differently from the previous ones, has a nonconstant Jacobian $J = 1 + 2\pi g \sin(2\pi y_n)$. In particular, when the parameter g is larger than $1/2\pi$, J vanishes along two horizontal lines and, simultaneously, the transformation is no longer invertible. Therefore, the theorem of L. S. Young,[33]

Table I. Dimension Function $D(\gamma)$ (Evaluated at Five Different γ Values), and Uniformity Factor λ for Four Maps and Two Flows.[a]

Maps	$\gamma=-2$	$\gamma=-1$	$\gamma=0$	$\gamma=1$	$\gamma=2$	λ
Henon[b]	1.23 ± 0.02	1.24 ± 0.01	1.26 ± 0.01	1.27 ± 0.01	1.28 ± 0.01	≈0.02
Zaslavskij[b]	1.54 ± 0.02	1.55 ± 0.01	1.55 ± 0.01	1.56 ± 0.01	1.57 ± 0.02	≈0.01
Sinai[b]	1.60 ± 0.01	1.66 ± 0.01	1.76 ± 0.01	1.88 ± 0.01	2.00 ± 0.01	≈0.10
Sinai[c]	1.57 ± 0.02	1.65 ± 0.01	1.73 ± 0.01	1.86 ± 0.01	2.01 ± 0.01	≈0.11
Flows	$\gamma=-1$	$\gamma=0$	$\gamma=1$	$\gamma=2$	$\gamma=3$	λ
Rössler[d]	2.90 ± 0.03	2.92 ± 0.03	2.93 ± 0.03	2.97 ± 0.03	3.01 ± 0.03	≈0.03
Franceschini[e]	3.10 ± 0.02	3.12 ± 0.02	3.13 ± 0.02	3.15 ± 0.01	3.16 ± 0.01	≈0.01

[a] Parameter values for the Henon and Zaslavskij attractors as in Refs. 18, 21, and 30, for the Roessler system as in Ref. 31 and, for the Sinai map, $g=0.3$. Note that the γ values are integers between -2 and 2 for the maps, and between -1 and 3 for the flows. See text for further information.

[b] 5000.2^{19} Points.

[c] 40000.2^{16} Points.

[d] 40000.2^{15} Points.

[e] 5000.2^{16} Points.

proving the equality between information dimension D_1 and Lyapunov dimension D_L, does not apply. Indeed, indicating with Λ_1 and Λ_2 the Lyapunov exponents, we found that $D_L=1+\Lambda_1/|\Lambda_2|$ is very close to D_1, as long as $g<1/2\pi$, but moves toward the value $D_0=2$, for $g>1/2\pi$. We also found that the attractor becomes highlyy nonuniform ($\lambda\approx0.1$) for $g=0.2$, 0.25, and 0.3 (all larger than $1/2\pi$), while it is very uniform for small g.

The Roessler system

$$\begin{aligned}
\dot{x} &= -y-z \\
\dot{y} &= x+0.25y+w \\
\dot{z} &= 2.2+xz \\
\dot{w} &= 0.5w-0.5z
\end{aligned} \tag{6.2}$$

exhibits two interesting features: nonconstant divergence and two positive Lyapunov exponents. The evaluation of $D(\gamma)$ is delicate because the variable z remains close to zero (≈0.1) for long times and, suddenly, jumps for short times up to values around 270. This region being very rarely visited, a large number of points is required to average over a representative sample of the attractor. Therefore, even choosing 40 000 points, the errors are larger than in other examples. The attractor does not seem to be very nonuniform ($\lambda \approx 0.03$) but, also in this case, the Lyapunov dimension D_L is closer to D_0, rather than to D_1.

The seven-mode system of Ref. 32 has been studied with the Reynolds number $R = 400$, where a strange attractor exists. With the following initial conditions ($x_1 = 0.264\,827\,9$, $x_2 = 1.564\,438\,7$, $x_3 = 5.544\,144\,0$, $x_4 = 2.687\,856\,0$, $x_5 = 8.678\,583\,8$, $x_6 = -0.997\,157\,2$, $x_7 = 8.431\,487\,7$), the Lyapunov dimension is $D_L = 3.12$. In contrast to the picture of the attractor, as obtained from a Poincaré section,[32] which exhibits a very dense region surrounded by more dilute ones, the uniformity factor is quite low. Also, the Kaplan–Yorke estimate is in agreement with the value $D(0)$.

Other models have been analyzed, like the Lorenz attractor[34] and the binary Cantor set recovering, in the first case, the known value $D(\gamma) \approx 2.06$ for any γ and, in the second one, the theoretical results $D_0 = 1$ and $D_1 = -(p_1 \ln p_1 + p_2 \ln p_2)/\ln 2$.

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APPENDIX A. DIMENSION FUNCTION FOR A NONRECURRENT SET OF POINTS

In this Appendix, we study the dimension function for the anomalous set of points $A = \{x \mid x = 1/n^\beta\}$. A random choice of points in this case is meaningless because it would always yield $x = 0$, which does not even belong to the set. A more sensible method would be to fix a rule to fill the set A. As already pointed out, different rules will change the curve $D(\gamma)$, except for the fixed point. Hence, here we follow the natural generation rule, namely, 1, $1/2^\beta$, $1/3^\beta$,..., $1/n^\beta$. In this way, we note that all these points are not recurrent. As the exclusion of a finite number of points does not affect the final result, we neglect the contribution of the first n_0 points. Thus, we can approximate $\delta_i(n)$ with

$$\delta_i(n) \approx \frac{\beta}{i(i+1)^\beta} \tag{A1}$$

and, substituting the sum over i with an integral,

$$\langle \delta^\gamma \rangle = \frac{\beta^\gamma}{n - n_0} \int_{n_0}^{n} \frac{di}{i^{\gamma(\beta+1)}} \tag{A2}$$

By recalling (3.2), we obtain, apart from multiplicative factors,

$$n^{-\gamma/D(\gamma)} \sim \begin{cases} \dfrac{\beta^\gamma}{1-\gamma(\beta+1)} [n^{-\gamma(\beta+1)} - n^{-1} n_0^{1-\gamma(\beta+1)}], & \gamma \neq \dfrac{1}{1+\beta} \\ \beta^\gamma \dfrac{\ln n}{n}, & \gamma = \dfrac{1}{1+\beta} \end{cases} \tag{A3}$$

Either the first or the second term on the right-hand side, asymptotically predominates, according to the value of γ. Namely, if $\gamma < 1/(1+\beta)$, $D(\gamma) = 1/(1+\beta)$ and, if $\gamma > 1/(1+\beta)$, $D(\gamma) = \gamma$ (see Fig. 5). The existence of an infinity of fixed points justifies the importance of taking the infimum in the definition of D_0, avoiding possible ambiguities. Moreover, at the fixed point, the generalized volume does not tend to a constant—instead it shows a logarithmic law.

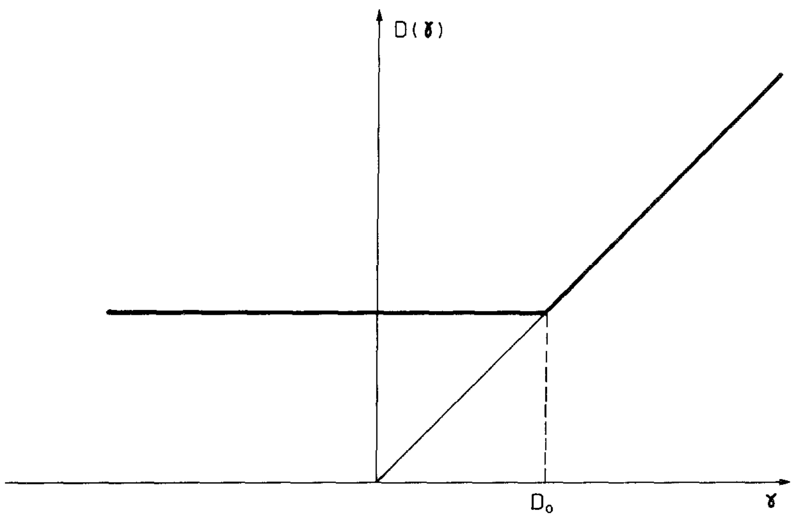

Fig. 5. Function $D(\gamma)$ for the example of Appendix A.

Following the above prescription, D_0 turns out to be

$$D_0 = 1/(1+\beta) \tag{A4}$$

This result coincides with the well-known value of the capacity, confirming the argued equivalence of the two quantities. The constancy of the left part of the curve (with respect to D_0) does not imply the uniformity of the set, as it is clearly evidenced by the right part. Note that the method is successful also in this peculiar example, where the nn distances do not tend to zero.

APPENDIX B. UNION OF DIFFERENT CANTOR SETS

Let us consider the union U of m uniform Cantor sets, each one generated by dividing the unit segment into a_i parts and deleting $(a_i - n_i)$ of them. Let us indicate with p_i the weight of the ith set ($\sum_{i=1}^{m} p_i = 1$). If n is the total number of points, np_i of them will belong to the ith Cantor set. This is tantamount to stating that the generation step is labeled by the index

$$k_i = \ln(np_i)/\ln(n_i) \tag{B1}$$

Hence, the np_i nn distances in the ith set are all equal to $a_i^{-k_i}$. The straightforward application of relation (3.2) yields

$$\langle \delta^\gamma \rangle = \sum_{i=1}^{m} p_i a_i^{-k_i \gamma} \tag{B2}$$

which can be written as

$$\langle \delta^\gamma \rangle = \sum_{i=1}^{m} p_i a_i^{-\gamma \ln p_i / \ln n_i} \cdot n^{-\gamma/\ln n_i / \ln a_i} \sim n^{-\gamma/D(\gamma)} \tag{B3}$$

Note that, when $\gamma > 0$, in the asymptotic limit $n \to \infty$, the leading term on the left-hand side is the one with the largest exponent $\ln(n_i)/\ln(a_i)$, independently of γ (see Fig. 6). Vice versa, when $\gamma < 0$, only the smallest exponent survives. It is then interesting to study the discontinuity point $\gamma = 0$, which corresponds to taking the geometric mean. The result is

$$D(0) = 1 \Big/ \sum_{i=1}^{m} p_i \frac{\ln a_i}{\ln n_i} \tag{B4}$$

Clearly, the capacity D_0 of U coincides with the largest one of its subsets, according to an obvious request for a definition of dimension. Moreover, it

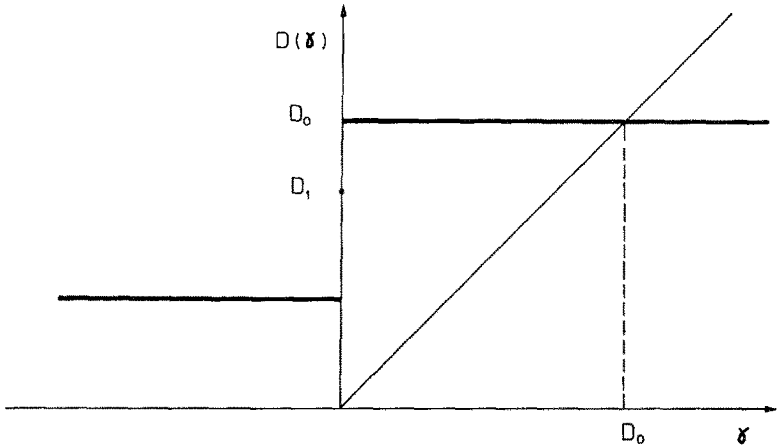

Fig. 6. Function $D(\gamma)$ for the superposition of different Cantor sets, Appendix B.

does not evidently depend on the weights p_i. The information dimension $D(0)$, on the contrary, is sensitive to all geometrical and probabilistic features of U. All the other Renyi D_q yield the smallest of the dimensions of the Cantor sets, thus showing that they are not able, in this example, to catch the relevant information for the ensemble U. Finally, it is evident that the uniformity factor being 0, it becomes useless, and the whole picture of the set is only provided by the complete DF.

APPENDIX C. nn DISTRIBUTION FOR THE BINARY CANTOR SET

A direct evaluation of $\bar{S}(\delta, n) = 1 - S(\delta, n)$ is developed here for a binary Cantor set. Let p_1 and p_2 be the probabilities of bits 0 and 1, respectively. The mass inside a generic segment of width $2\delta = 2^{-k}$ is simply given by the probability $p_1^i p_2^{k-i}$, where i is nothing but the number of 0's among all the k bits defining the segment given. Hence, the probability $\bar{S}$ of having no points inside such a segment after n "throws" is

$$\bar{S} = \sum_{i=0}^{k} (1 - p_1^i p_2^{k-i})^n \binom{k}{i} p_1^i p_2^{k-i} \tag{C1}$$

where $\binom{k}{i}$ indicates the multiplicity of each sequence of bits. We now study the asymptotic limit of large n and, consequently, small δ's. The first factor on the right-hand side can be approximated with an exponential, the rest with a Gaussian centered around kp_1 having variance $kp_1 p_2/2$, and the sum goes into an integral

$$\bar{S} = \frac{1}{(2\pi k p_1 p_2)^{1/2}} \int_0^k \exp\left[-n\left(\frac{p_1}{p_2}\right)^i p_2^k\right] \exp\left[-\frac{(i-kp_1)^2}{kp_1 p_2}\right] di \tag{C2}$$

It is, then, convenient to introduce the variable $f = i/k$ and substitute $2\delta = 2^{-k}$; hereafter, we also indicate with log the base-of-two logarithm

$$\bar{S} = \left[\frac{-\log(2\delta)}{2\pi p_1 p_2}\right]^{1/2} \int_0^1 \exp(-n(2\delta)^{-[f \log p_1 + (1-f)\log p_2]}) \times \exp\left[\log 2\delta \frac{(f-p_1)^2}{p_1 p_2}\right] df \tag{C3}$$

To make a clear comparison with the uniform case, we introduce the variable

$$\mathscr{D} = -f \log p_1 - (1-f) \log p_2 \tag{C4}$$

yielding

$$\bar{S} = \left[\frac{-\log(2\delta)}{2\pi p_1 p_2}\right]^{1/2} \frac{1}{\log p_1/p_2} \int_{-\log p_1}^{-\log p_2} \exp[-n(2\delta)^{\mathscr{D}}] \times \exp\left[\frac{\log(2\delta)}{p_1 p_2 \log^2(p_2/p_1)} (\mathscr{D} + p_1 \log p_1 + p_2 \log p_2)^2\right] d\mathscr{D} \tag{C5}$$

where we have assumed $p_2 < p_1$. This expression can be interpreted as a superposition of distributions of uniform Cantor sets with different dimensions. Note that $\mathscr{D}$ ranges between $-\log p_1$ and $-\log p_2$ which coincide with the asymptotes of the DF $D(-\infty)$ and $D(\infty)$, as can be ascertained from (3.7) taking $\alpha_1 = \alpha_2 = 2$.

Moreover, the weight function is a Gaussian centered at the information dimension which, then, appears to play a special role.

An analytic expression for the integral (C5) is not available, but its interpretation is fully contained in the interplay of the two exponential functions.

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